4. RELATIONS AND FUNCTIONS

§4.1. Relations

Having defined relations and functions as certain types of statement we redefine them within the class of sets. The ones built up from statements of the form $x \in y$ we will now call 'generalized' relations and functions.

A **relation** R between sets X and Y is a subset of X × Y. We denote the statement that $(x, y) \in$ R by ***x*R*y*** and denote the fact that R is a relation between X and Y by **R:X→Y**. A relation between a set X and itself is called a **relation on X**.

For example the relation $m < n$ on ℕ, the set of natural numbers, would be considered to be the set of all pairs (m, n) where $m < n$. Of course you must be aware that, although we are talking about natural numbers and their ordering, by way of an example, we have yet to define these properly.

This is the way most relations are stored in a computer database. The relation xEy, where x is a student and y is a course in some university, might be defined to mean that x is enrolled in course y. There is no algorithm for deciding whether a given student is enrolled in a certain course. Instead this relation may be stored as an array of ordered pairs (student, course).

We define **multiplication** of relations as follows. If R:X→Y and S:Y→Z are relations then **RS**:X→Z is defined by x(RS)z if there exists $y \in$ Y such that xRy and ySz. In symbols this would be x(RS)z $\leftrightarrow \exists y[x\mathrm{R}y \wedge y\mathrm{S}z]$.

Example 1: Suppose a = Alice, b = Bill and c = Catherine and suppose that Bill and Catherine are Mary's two children. If P is the relation 'parent of' and S is the relation 'sibling of' then we would have aPb and aPc and bSc and cSb.

The relation SP is the relation 'aunt or uncle of'. What about PS? A parent of my sibling would be my sibling. At least that would be the case if there were no step mothers, step fathers or step children. So is PS = P? Well no, if I am an only child, my father is one of my parents but he would not be a parent of my sibling. As sets of ordered pairs PS would be a subset of P.

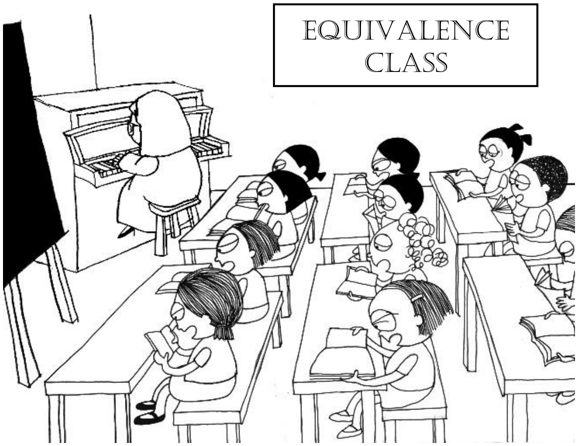
The **inverse** of a relation R: X→Y is $\mathbf{R^{-1}}$:Y→X defined by $x\mathrm{R}^{-1}y$ if and only if $y\mathrm{R}x$.
In other words, $\mathrm{R}^{-1} = \{(y, x) \mid (x, y) \in \mathrm{R}\}$.

Example 2: If P and S are as in Example 1 then P^{-1} is the relation of being 'child of' while $\mathrm{S}^{-1} = \mathrm{S}$. If Ron is my father then I am a child of Ron's. And I am a sibling to any of my siblings.

A relation R on a set X is:
- **reflexive** if $\forall x[x\mathrm{R}x]$,
- **symmetric** if $\forall x \forall y[\underline{x\mathrm{R}y} \rightarrow y\mathrm{R}x]$,
- **transitive** if $\forall x \forall y \forall z[x\mathrm{R}y \wedge y\mathrm{R}z \rightarrow x\mathrm{R}z]$.
- an **equivalence relation** if it is reflexive, symmetric and transitive.

If R is an equivalence relation, the **equivalence class** containing x is $\mathbf{[x]_R} = \{y \mid x\mathrm{R}y\}$. Often we leave out the subscript and simply write $[x]$. If $y \in [x]$ then $[y] = [x]$. Distinct equivalence classes are disjoint since $z \in [x] \cap [y]$ implies that $x\mathrm{R}z$ and $y\mathrm{R}z$ and together these imply $x\mathrm{R}z$.

Hence an equivalence relation on a set partitions the set into disjoint equivalence classes.



Example 3: The relation P in Examples 1 and 2 is neither reflexive nor symmetric. Nobody can be their own father (reflexive) nor am I my father's father (symmetric). And clearly P isn't transitive.

The normal use of the word 'sibling' precludes anyone from being their own sibling, so S is not reflexive. But, as we have seen, S is symmetric. Is S transitive? Is my sibling's sibling my own sibling? Not quite. If b, c are Bill and Catherine in Example 1, then $b\mathrm{S}c$ and $c\mathrm{S}b$. If S was transitive it would follow that $b\mathrm{S}b$. But Bill is not his own sibling. Remember that in the definition of the transitive property nothing was said about the x, y and z being distinct.

Example 4: Show that the relation xRy define on the set of integers by xRy if $x - y$ is a multiple of 7, is an equivalence relation.

Solution: Reflexive: For all x, $x - x = 0$, which is a multiple of seven.

Symmetric: Suppose xRy. Then $x - y$ is a multiple of 7. Hence $y - x$ is a multiple of 7 and so yRx.

Transitive: Suppose xRy and yRz.

Then $x - y$ and $y - z$ are multiples of 7.

Hence $x - z = (x - y) + (y - z)$ is a multiple of 7 and so xRz. So R is an equivalence relation.

The set $\{\ldots, -12, -5, 2, 9, \ldots\}$ is an example of an equivalence class for this equivalence relation since it consists of all numbers that are equivalent to 2.

A relation R on a set X is **regular** if $\forall x \exists y [x \mathrm{R} y]$.

Example 5: If R is the relation $x < y$ on $\mathbb{N}$ then R is regular. ($n < n + 1$), But if R is $x > y$ then it is not regular because 0 is not greater than any natural number.

Theorem 1: If R is a regular relation then it is an equivalence relation if and only if

$$\mathrm{R}^{-1} = \mathrm{R} \text{ and } \mathrm{R}^2 = \mathrm{R}.$$

Proof: The statement $\mathrm{R}^{-1} = \mathrm{R}$ is equivalent to the symmetric property and $\mathrm{R}^2 \subseteq \mathrm{R}$ is equivalent to the transitive property.

Suppose that R is an equivalence relation and let $x \in X$. Suppose $(x, y) \in R$, in other words, xRy. Then, since xRx and xRy, it follows that xR^2y and so $(x, y) \in R^2$.

Hence $R^2 = R$.

Suppose now that R is a regular relation and that

$$R^{-1} = R \text{ and } R^2 = R.$$

Then R is clearly symmetric and transitive. All that remains to show is that R is reflexive.

Let $x \in X$. Since R is regular xRy for some $y \in Y$. Since R is symmetric yRx and since R is transitive xRx and so R is reflexive.

Example 6: There are 7 equivalence classes in example 4. For example [3] = {..., –18, –11, –4, 3, 10, 17, 24, ...}

§4.2. Functions

Most of the material in the next couple of paragraphs should be well known to you and so I will be brief.

A **function** $F:X\rightarrow Y$ is a relation between X and Y such that: $\forall x\forall y\forall z[xFy \wedge xFz \rightarrow y = z]$. The unique y is denoted by $F(x)$. As a relation, it is a subset of $X \times Y$.

If $F:X\rightarrow Y$ is a function, X is called the **domain** and Y is the **codomain**.

The **image** (**range**) of F is **im** F = $\{y \mid \exists x[y = F(x)]\}$.

The function $F:X\to Y$ is **1-1** (**injective**) if:

$$\forall x\forall y[F(x) = F(y) \to x = y].$$

It is **onto** (**surjective**) if im F = Y.

A function is a **bijection** if it is 1-1 and onto.

A function F is **invertible** if $F^{-1}:Y\to X$ is a function and so $F: X\to Y$ is invertible if and only if it is a bijection.

A **permutation** on a set X is a bijection $F: X\to X$. The inverse of a permutation is a permutation and the product of two permutations is a permutation.

Clearly if F, G are 1-1 then FG is 1-1. If F, G are onto then FG is onto.

If $F:X \to Y$ and $Z \subseteq X$ then $F|Z = F \cap (Z \times Y)$ (the **restriction** of F to Z).

The product of two functions is defined as for relations. If $F:X\to Y$ and $G:Y\to Z$ then the **product** $FG:X\to Z$ is defined by: $(FG)(x) = G(F(x))$. This is the same as if we considered them as relations.

We denote the set of all functions $F:Y \to X$ by $\mathbf{X^Y}$. This notation is motivated by the fact that if X and Y are finite sets with sizes *m* and *n* respectively then X^Y has size

m^n. This is because there are m choices for F(y) for each of the n elements $y \in$ Y.

Theorem 2: If X, Y are sets then so is Y^X.
Proof: A typical element of X^Y is a function F: X→Y.
It is a set of ordered pairs of the form (x, F(x)) for some
$x \in$ X.
But (x, F(x)) = {{x}, {x, F(x)}}.
Now $x \in$ X and F(x) $\in$ Y, so both belong to X ∪ Y.
Hence {x} and {x, F(x)} are subsets of X ∪ Y, so they're both elements of ℘(X ∪ Y).
So far, all of these are sets by various ZF axioms.
Hence (x, y) = {{x}, {x, F(x)}}
and so is a subset of ℘(X ∪ Y).
Therefore (x, F(x)) $\in \wp^2$(X ∪ Y).
Now such an F is a set of these so F is a subset of $\wp^2$(X ∪ Y) and hence is an element of $\wp^3$(X ∪ Y).
Finally, X^Y is a set of these functions and so is a subset of $\wp^3$(X ∪ Y). Thus $X^Y \in \wp^4$(X ∪ Y). By the axioms of unions and power sets $\wp^4$(X ∪ Y) is a set and by the Axiom of Specification we can prove that X^Y is a set.

Clearly $\varnothing^X = \varnothing$ if X ≠ ∅ since there are no functions from a non-empty set to the empty set. But $X^\varnothing = \{\varnothing\}$ for all X.

§4.3. *n*-Tuples, Sequences and Families

We have yet to define numbers. That will have to wait until the next chapter. Here we will consider certain mathematical objects that we use in mathematics, such as *n*-tuples, sequences and matrices.

We have already seen how to view an ordered pair as a set: $(a, b) = \{\{a\}, \{a, b\}\}$. It would be possible to extend this to an ordered triple, or a general *n*-tuple, but it's more convenient to do this by means of functions from sets of natural numbers. Never mind that we don't have any such objects at this stage.

We denote the set $\{1, 2, \ldots, n\}$ by **[*n*]**.
We define an ***n*-tuple** as a function, x: $[n] \rightarrow$ S, for some set S. We usually denote $x(k)$ by x_k and the *n*-tuple itself as $(x_1, x_2, \ldots, x_k)$. We call $[n]$ the **indexing set**. Since functions from sets to sets are sets then *n*-tuples are sets. And the class of all *n*-tuples on a set S is simply $S^{[n]}$ and so, by Theorem 2, it is itself a set. You may have noticed that usually $S^{[n]}$ is usually written as $S \times S \times \ldots \times S$ (*n* factors), or more simply as S^n.

Is the ordered pair (a, b) also a 2-tuple? Well, yes and no. In the course of doing ordinary mathematics we would treat them as the same thing, and that's fine. But we defined ordered pairs separately to *n*-tuples and, if we drill down to their nature as sets, they will be different.

The ordered pair (a, b) was defined as:

$$\{\{a\}, \{a, b\}\}.$$

But the 2-tuple (a, b) is the function $F:\{1, 2\}\to S$ where $F(1) = a$ and $F(2) = b$. Now viewing this function as a relation it would be $\{(1, a), (2, b)\}$ and writing each of these ordered pairs as a set, we would have:

$$(a, b) = \{\{\{1\}, \{1, a\}\}, \{\{2\}, \{2, b\}\}\}.$$

Why didn't we define *n*-tuples first and then automatically we would have had ordered pairs as 2-tuples. Can you see why this wouldn't work? Our definition of *n*-tuples was in terms of functions and functions were defined in terms of ordered pairs. So we were forced to define ordered pairs first, before *n*-tuples!

But the distinction between an ordered pair and a 2-tuple is purely a technical one in the context of setting mathematics on the foundations of set theory. In everyday mathematics the distinction is artificial and is ignored.

Often we have sequences where the components come from different sets. The Cartesian product $R \times S \times T$ consists of 3-tuples (x, y, z) where $x \in R$, $y \in S$ and $z \in T$. We would consider $R \times S \times T$ as a subset of $(R \cup S \cup T)^3$.

An $m \times n$ matrix over a set S can be considered as a function a from $[m] \times [n]$ to S, where $a(i, j)$ is usually written as a_{ij}.

When the number of components is infinite we use different terminology. We don't refer to an ∞-tuple, but rather as an **infinite sequence**. The infinite sequence:
$x_1, x_2, \ldots$ would be defined as a function from $\{1, 2, \ldots\}$ to a set S. Often it's convenient to begin the sequence with an x_0 (it makes no difference in practice) and then we can say that the infinite sequence $x_0, x_1, x_2, \ldots$ is simply a function from $\mathbb{N}$, the set of natural numbers, to the set S.

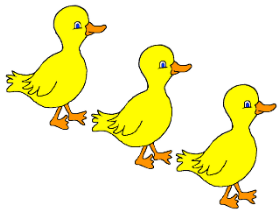
But we are 'jumping the gun' a little here because we haven't yet defined the natural numbers. We'll do that shortly.

Occasionally we need something bigger than a sequence. If I and S are sets we define a **family** $(F_i)_{i\in I}$ to be a function $F:I\rightarrow S$. The set I is called the **indexing set**. Here F_i is an alternative way of writing $F(i)$.

If $I = \{1, 2, \ldots, n\}$ this is simply an n-tuple.
If $I = \mathbb{N}$ then it's an infinite sequence. But we can consider more general families.

Example 7: In topology we consider systems of neighbourhoods of points. If $a \in \mathbb{R}^2$, considered as a point in the plane, and $r > 0$ we define the r-neighbourhood to be $N_r(a) = \{x \mid |x - a| < r\}$, an open circle of radius r centred at a. These form a family $(N_r(a))_{r\in\mathbb{R}^+}$ where $\mathbb{R}^+$ is the indexing set.

With all our work so far we still cannot even talk about kindergarten arithmetic. Counting 1, 2, 3, … exposes the young child to a very abstract concept. What exactly is the number '3'? We point to a picture of three ducks and a picture of three balls and eventually the child manages to extract the 'three-ness'. But even mature adults would generally be unable to give a satisfactory definition of 'three'. Fortunately we get along in life without having formal definitions. How would you define a cat, for example?



But we've undertaken the job of establishing mathematics on a rigorous foundation, so we're going to have to define numbers precisely. That's our very next task.